Ma 350: Mathematics for Multimedia  
Homework Assignment 6  
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1. Let \( w \) be the Haar mother function defined by Equation 5.2. Prove that the set of functions \( f_k : k \in \mathbb{Z} \) defined by \( f_k(t) = w(t - k) \) is orthonormal.

2. Show that if \( h = \{ h(k) : k \in \mathbb{Z} \} \) is a self-orthonormal filter, and \( M \) is any fixed integer, then the sequence defined by

\[
g(k) = (-1)^k h(2M - 1 - k), \quad \text{for all} \ k \in \mathbb{Z},
\]

satisfies the completeness condition of Equation 5.45.

3. a. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the antisymmetry condition \( h(0) = -h(3) \) and \( h(1) = -h(2) \)?

b. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the symmetry condition \( h(0) = h(3) \) and \( h(1) = h(2) \)?

4. Suppose that an orthogonal MRA has a scaling function \( \phi \) satisfying \( \phi(t) = 0 \) for \( t \notin [a, b] \). Prove that the low-pass filter \( h \) for this MRA must satisfy \( h(n) = 0 \) for all \( n \notin [2a - b, 2b - a] \). (This makes explicit the finite support of \( h \) in Equation 5.36.)

5. Suppose that \( x, y, a, b \) are integers with \( x \geq y \) and \( b \geq a \). Let \( u = \{ u(k) : k \in \mathbb{Z} \} \) be a sequence supported in \( [x, y] \) and let \( f = \{ f(k) : k \in \mathbb{Z} \} \) be a filter sequence supported in \( [a, b] \) that defines a filter transform \( F \) and its adjoint \( F^* \) as in Equations 5.61 and 5.62. What is the support interval for \( FF^*u \)? What if \( f \) satisfies the self-orthonormality condition?

6. Suppose that \( h = \{ h(k) : k \in \mathbb{Z} \} \) and \( g = \{ g(k) : k \in \mathbb{Z} \} \) satisfy the orthogonal CQF conditions. Show that the 2-periodizations \( h_2, g_2 \) of \( h \) and \( g \) are the Haar filters. Namely, show that \( h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2} \).

7. Let \( \phi \) be the scaling function of an orthogonal MRA, and let \( \psi \) be the associated mother function. For \( (x, y) \in \mathbb{R}^2 \), define

\[
\begin{align*}
e_0(x, y) &= \phi(x)\phi(y), & e_1(x, y) &= \phi(x)\psi(y) \\
e_2(x, y) &= \psi(x)\phi(y), & e_3(x, y) &= \psi(x)\psi(y).
\end{align*}
\]

Prove that the functions \( \{ e_n : n = 0, 1, 2, 3 \} \) are orthonormal in \( L^2(\mathbb{R}^2) \), the inner product space of square-integrable functions on \( \mathbb{R}^2 \).
8. Fix an integer \( N > 1 \) and consider a graph with vertices labeled \( 1, \ldots, N \) with each pair of vertices connected by an edge. Compute the total number of edges, and list them.

9. Construct a prefix code for the alphabet \( A = \{a, b, c, d\} \) with codeword lengths 1,2,3,4, or prove that none exists.

10. Construct a prefix code for the 26-letter English alphabet \( A = \{a, b, c, \ldots, z\} \) with longest codeword 4, or prove that none exists.

11. Suppose we have two prefix codes, \( c_0(a, b) = (1,0) \) and \( c_1(a, b) = (0,1) \), for the alphabet \( A = \{a, b\} \). Show that the following dynamic encoding is uniquely decipherable by finding a decoding algorithm:

   **Simple Dynamic Encoding Example**

   ```
   dynamicencoding0( msg[], M ):
   [0] Initialize n=0
   [2] Transmit msg[m] using code n
   [3] If msg[m]=='b', then toggle n = 1-n
   ```

   (This encoding is called dynamic because the codeword for a letter might change as a message is encoded, in contrast with the static encodings studied in this chapter. It gives an example of a uniquely decipherable and instantaneous code which is nevertheless not a prefix code.)